

Two-loop self-energy diagrams worked out with NDIM

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Abstract. In this work we calculate two two-loop massless Feynman integrals pertaining to self-energy diagrams using NDIM (Negative Dimensional Integration Method). We show that the answer we get is 36-fold degenerate. We then consider special cases of exponents for propagators and the outcoming results compared with known ones obtained via traditional methods.

1 Introduction

The dimensionality of space-time plays a key role in all branches of Physics. The quantities we calculate, as theoreticians, depend very much on the number of dimensions we are considering. Theories in higher and lower dimensions than four have been put forth by many researchers and plentiful of good insights have been gained through this exercise. Zooming in the arena of quantum field theory, we discover that the dimensionality of space-time gained a more sophisticated status, being promoted from a mere integer number to that of a complex variable, with the advent and development of dimensional regularization by 't Hooft et al. and several other pioneers in the field [1].

In other words, we could say that quantum field theory (QFT), besides other great ideas it inspired, physical and mathematical alike, did reveal this amazing possibility: the analytic continuation of the space-time dimension D .

The union between the theory of analytic functions and QFT is very profitable. Dimensional regularization (DREG), the technique that bears the concept of analytically continued D , is one of its profits. As a step further in this direction Halliday et al. [2,3] developed the idea of analytically continued D to *negative values*. Of course, the seminal idea of negative values for D is already contained in the work of 't Hooft and others. But what is novel in Halliday's insight is the amazing possibility of letting field propagators be raised to positive powers, so that the integrand becomes polynomial. The thrust behind the idea is that solving a polynomial integral should be – in principle at least – easier to perform than rational ones elicited in the usual Feynman integrals. This very simple argument, which we call negative dimensional integration method (NDIM), can simplify the calculation of Feynman integrals in an astounding way [4–10].

In the usual DREG [11–13] the only quantities that preserve their meaning are the Green's functions [14]. We will not try to discuss whether they still have (or have not) their meaning preserved within the context of NDIM

nor speculate what are the features, if any, of this “new world” of negative dimensions [8]. What we do is simply to allow for it just for calculational purposes. The reader must have this important point in mind.

In our previous works [7,8] we calculated massive one-loop four point functions (former reference) and a massless two-loop three-point vertex (latter reference) with the NDIM approach. In the first, NDIM provided not only the well-known hypergeometric functions but six other new results in a very straightforward manner [9]; while for the two-loop vertex, we considered the particular case where two of its external momenta were set on-shell, and NDIM responded with as many as twelve times – surprisingly enough, all of them yielding the same correct result – that is, a twelvefold degeneracy. This led us to conjecture that when the power series had unit argument and they were all summable, then the result would be degenerate. That is, if this conjecture is correct, we need only to carry out one sum – the most convenient one, of course. The conjecture remains to be proven or disproved.

Here we put our NDIM to another “lab-test” [8] by considering two two-loop self-energy diagrams which we call by the funny name “flying saucer” diagrams – side view (Fig. 1) and front view (Fig. 2), just to make it easier for us to refer to them. The outline for this article is as follows: In Sect. 2 we solve the two-loop Feynman integral relative to these two graphs, i.e., the space-time dimension and the exponents of the pertinent propagators are left arbitrary. Then, in Sect. 3, we particularize to suit either the “flying saucer, side view” or the “flying saucer, front view” diagram cases. And finally, Sect. 4 is devoted to our concluding remarks.

2 Feynman graphs with four massless propagators

The NDIM approach to solve Feynman integrals is beautiful in its simplicity: First, we take the propagators of the integral we want to work out, multiply each one of them

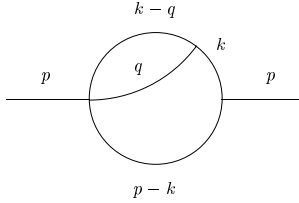


Fig. 1. Two-loop massless Feynman diagram: the “flying saucer”, side view

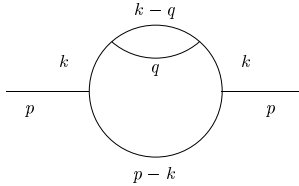


Fig. 2. Two-loop massless Feynman diagram: the “flying saucer”, front view

by a specific parameter and then solve the D -dimensional gaussian integral whose argument is that very expression. Let us see how it works in practice. Consider the gaussian integral,

$$I(p^2; D) = \int d^D k d^D q \exp \left[-\alpha q^2 - \beta k^2 - \gamma(p-k)^2 - \omega(k-q)^2 \right], \quad (1)$$

which clearly is relevant to the diagrams we want to work out. It is not difficult to integrate it; the result is,

$$I(p^2; D) = \left(\frac{\pi^2}{\lambda} \right)^{D/2} \exp \left[-\frac{\gamma p^2}{\lambda} (\beta\omega + \alpha\beta + \alpha\omega) \right], \quad (2)$$

where $\lambda = \alpha\beta + \beta\omega + \alpha\gamma + \gamma\omega + \alpha\omega$. Expanding (2) in Taylor series and also expanding the multinomial expression in λ , we get an eightfold summation,

$$I(p^2; D) = \sum_{n_i=0}^{\infty} \frac{(-p^2)^{n_1+n_2+n_3} (-n_1 - n_2 - n_3 - \frac{1}{2}D)!}{n_1!n_2!n_3!n_4!n_5!n_6!n_7!n_8!} \times \alpha^{n_1+n_2+n_4+n_6+n_8} \beta^{n_1+n_3+n_4+n_5} \times \gamma^{n_1+n_2+n_3+n_6+n_7} \omega^{n_2+n_3+n_5+n_7+n_8}, \quad (3)$$

with the constraint $-n_1 - n_2 - n_3 - \frac{1}{2}D = n_4 + n_5 + n_6 + n_7 + n_8$ coming from the multinomial expansion.

The second step is simpler and faster: expand the exponential (1) in Taylor series first to get,

$$I(p^2; D) = \sum_{i,j,l,m=0}^{\infty} (-1)^{i+j+l+m} \frac{\alpha^i \beta^j \gamma^l \omega^m}{i!j!l!m!} \times \mathcal{J}(i, j, l, m; D), \quad (4)$$

where we define,

$$\mathcal{J}(i, j, l, m; D) = \int d^D k d^D q (q^2)^i (k^2)^j [(p-k)^2]^l \times [(k-q)^2]^m, \quad (5)$$

which is our negative dimensional integral. Observe that the propagators are raised to *positive* powers. We will solve this integral and then carry out the analytic continuation of the result to allow *negative* powers of the propagators and *positive* D . This is a general feature of NDIM.

Comparing (3) and (4) we get an expression for the negative- D integral,

$$\mathcal{J}(i, j, l, m; D) = \sum_{n_i=0}^{\infty} \frac{(-\pi)^D (p^2)^\sigma G}{n_1!n_2!n_3!n_4!n_5!n_6!n_7!n_8!}, \quad (6)$$

where we define the product of gamma functions,

$$G = \Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1+m)\Gamma(1-\sigma - \frac{1}{2}D),$$

and since the two expressions must equal, sum indices in the former and exponents of propagators in the latter, must satisfy the system,

$$\begin{cases} n_1 + n_2 + n_4 + n_6 + n_8 = i \\ n_1 + n_3 + n_4 + n_5 = j \\ n_1 + n_2 + n_3 + n_6 + n_7 = l \\ n_2 + n_3 + n_5 + n_7 + n_8 = m \\ n_1 + n_2 + n_3 = \sigma \end{cases} \quad (7)$$

where $\sigma = i+j+l+m+D$ and the last equation comes from the multinomial expansion. Observe that the equations above are linear, but because we have eight unknowns and only five equations, in order to solve this system we must choose three of the unknowns and solve it in terms of them. There are many ways in which this choice can be done; in fact, there are $C_3^8 = 8!/(5!3!) = 56$ possibilities altogether. However, 20 out of the 56 lead us to trivial solutions, which present no interest at all. The remaining 36 give us the results for the Feynman integral when we plug their solutions in equation (6).

We will solve the non-trivial systems and write down the general results, but before doing that, let us see what we can do to lessen our task. Looking at the Feynman diagram we can spot symmetry properties that help us in this. Thus, we expect the outcoming result to be symmetric under the exchange $i \leftrightarrow m$, which in turn will reduce by half the number of distinct systems that we need to deal with, since the symmetry will account for the remaining half.

Let us then first consider the solution that leaves n_5, n_6, n_8 as free indices in the summation; let us call it \mathcal{J}_a . It yields,

$$\begin{aligned} \mathcal{J}_a &= (-\pi)^D (p^2)^\sigma P_1 \sum_{n_5, n_6, n_8=0}^{\infty} \frac{(-1)^{n_6} (-i - m - \frac{1}{2}D |n_8)}{n_5!n_6!n_8!} \\ &\times \frac{(-l + \sigma |n_6) (-j - m - \frac{1}{2}D |n_5 - n_6)}{(1 + l - m |n_5 - n_6 + n_8)} \\ &\times (\frac{1}{2}D + l |n_5 + n_8), \end{aligned} \quad (8)$$

where

$$P_1 = \frac{\Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1+m)}{\Gamma(1+l-m)\Gamma(1+i+m + \frac{1}{2}D)}$$

$$\times \frac{\Gamma(1 - \sigma - \frac{1}{2}D)}{\Gamma(1 + j + m + \frac{1}{2}D)\Gamma(1 + l - \sigma)\Gamma(1 - l - \frac{1}{2}D)},$$

with the Pochhammer symbol [15] denoted by,

$$(a|m) \equiv (a)_m = \frac{\Gamma(a + m)}{\Gamma(a)}.$$

Using one of the properties of the Pochhammer symbol [15], i.e.,

$$(a| - m) = \frac{(-1)^m}{(1 - a|m)}, \quad (9)$$

one can identify these series as hypergeometric [16]; in fact, we can rewrite them in a convenient manner using another property,

$$(a|b + c) = (a|b)(a + b|c), \quad (10)$$

and sum, for example, the n_8 series using the well-known formula [15],

$${}_2F_1(a, b; c|1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \quad (11)$$

yielding,

$$\begin{aligned} \mathcal{J}_a &= (-\pi)^D (p^2)^\sigma P_1 P_2 \sum_{n_5, n_6=0}^{\infty} \frac{(1 + i| - n_6)(-l + \sigma|n_6)}{n_5! n_6!} \\ &\times \frac{(\frac{1}{2}D + l|n_5)(-j - m - \frac{1}{2}D|n_5 - n_6)}{(1 - m - \frac{1}{2}D| - n_6)} \\ &\times \frac{1}{(1 + i + l + \frac{1}{2}D|n_5 - n_6)}, \end{aligned} \quad (12)$$

where

$$P_2 = \frac{\Gamma(1 + i)\Gamma(1 + l - m)}{\Gamma(1 - m - \frac{1}{2}D)\Gamma(1 + i + l + \frac{1}{2}D)}. \quad (13)$$

In a similar manner we can sum the two remaining series, getting as a result,

$$\begin{aligned} \mathcal{J}_a &= (-\pi)^D (p^2)^\sigma \frac{\Gamma(1 + i)\Gamma(1 + l)\Gamma(1 + m)}{\Gamma(1 + \sigma)\Gamma(1 - i - \frac{1}{2}D)\Gamma(1 - m - \frac{1}{2}D)} \\ &\times \frac{\Gamma(1 - \sigma - \frac{1}{2}D)\Gamma(1 + i + j + m + \frac{1}{2}D)}{\Gamma(1 - l - \frac{1}{2}D)\Gamma(1 + i + m + \frac{1}{2}D)} \\ &\times \frac{\Gamma(1 - i - m - D)}{\Gamma(1 + l - \sigma)}. \end{aligned} \quad (14)$$

The last and final step that need to be taken is to bring this result back to our real physical world with positive D . Grouping the gamma functions in the numerator with the ones in the denominator in convenient Pochhammer symbols and using (9), we arrive at

$$\begin{aligned} \mathcal{J}_a^{AC} &= \pi^D (p^2)^\sigma (-i|i + m + \frac{1}{2}D)(-m|i + m + \frac{1}{2}D) \\ &\times (\sigma + \frac{1}{2}D| - 2\sigma - \frac{1}{2}D)(-i - j - m - \frac{1}{2}D|j) \\ &\times (-l|\sigma)(i + m + D| - i + l - m - \frac{1}{2}D). \end{aligned} \quad (15)$$

This very simple operation allows us to analytically continue the result back into our real physical world, $D > 0$. Equation (15) is the general result, and we note that it is symmetric in $i \leftrightarrow m$ as it should be, and it is correct [17]. The reader will ask *immediately*: What is(are) the result(s) that the other solution(s) provide? NDIM answers in as brief and surprising a manner as it could be: *the same*. Indeed just to make sure we went through all of them, and verified that it is possible to sum all the emerging series and they provide the same result, namely, equation (14) which leads to the correct expression, equation (15), i.e., we have a thirty-six-fold degeneracy!

A word of caution here: Not all the sums can be so easily dealt with. Yet, just to convince the reader that it is possible to sum them all, we shall carry out one more summation, the hardest one. The degeneracy above mentioned can be classified into two sets: 32 solutions are like \mathcal{J}_a with relatively easy sums to carry out, while 4 of them are like the following one which we call \mathcal{J}_b . Consider then the solution with indices n_1, n_4, n_5 ,

$$\begin{aligned} \mathcal{J}_b &= (-\pi)^D (p^2)^\sigma P_3 \sum_{n_1, n_4, n_5=0}^{\infty} \frac{(\frac{1}{2}D + l|n_4 + n_5)}{n_1! n_4! n_5!} \\ &\times \frac{(-1)^{n_4} (\frac{1}{2}D + m|n_1 + n_4)(-j|n_1 + n_4 + n_5)}{(1 - j + \sigma|n_4 + n_5)(1 - i - j - \frac{1}{2}D|n_1 + n_4)}, \end{aligned} \quad (16)$$

where

$$\begin{aligned} P_3 &= \frac{\Gamma(1 + i)\Gamma(1 + l)\Gamma(1 + m)\Gamma(1 - \sigma - \frac{1}{2}D)}{\Gamma(1 - j + \sigma)\Gamma(1 - m - \frac{1}{2}D)\Gamma(1 - i - j - \frac{1}{2}D)} \\ &\times \frac{1}{\Gamma(1 - l - \frac{1}{2}D)}. \end{aligned} \quad (17)$$

Using the same procedure we used for \mathcal{J}_a , we can carry out the n_5 summation, yielding,

$$\begin{aligned} \mathcal{J}_b &= (-\pi)^D (p^2)^\sigma P_3 P_4 \sum_{n_1, n_4=0}^{\infty} \frac{(\frac{1}{2}D + l|n_4)(-\sigma|n_1)}{n_1! n_4!} \\ &\times \frac{(\frac{1}{2}D + m|n_1 + n_4)(-j|n_1 + n_4)}{(\frac{1}{2}D + l - \sigma|n_1 + n_4)(1 - i - j - \frac{1}{2}D|n_1 + n_4)}, \end{aligned} \quad (18)$$

where

$$P_4 = \frac{\Gamma(1 - j + \sigma)\Gamma(1 - l - \frac{1}{2}D + \sigma)}{\Gamma(1 + \sigma)\Gamma(1 - j - l - \frac{1}{2}D + \sigma)}.$$

However, this time, neither of the remaining sums (in n_1 or n_4) can be written in terms of ${}_2F_1$, but rather in terms of ${}_3F_2$ which are not summable for arbitrary values of its parameters. Here comes the trick that will do the required job: Put the n_4 series in terms of a ${}_3F_2$ function and use the property [16]

$${}_3F_2(a, b, c; e, f|1) = Q {}_3F_2(e - a, f - a, s; s + b, s + c|1), \quad (19)$$

where $s = e + f - a - b - c$ and

$$Q = \frac{\Gamma(e)\Gamma(f)\Gamma(s)}{\Gamma(a)\Gamma(s + b)\Gamma(s + c)}.$$

A good choice is to take

$$\begin{aligned} a &= -j + n_1 \\ b &= \frac{1}{2}D + l \\ c &= \frac{1}{2}D + m + n_1 \\ e &= \frac{1}{2}D + l - \sigma + n_1 \\ f &= 1 - i - j - \frac{1}{2}D + n_1 \end{aligned} \quad (20)$$

so that the gamma functions in Q simplify several factors in the series and some of them can be grouped by the property (10) giving,

$$\begin{aligned} \mathcal{J}_b &= (-\pi)^D (p^2)^\sigma P_3 P_4 P_5 \\ &\times \sum_{n_1, n_4=0}^{\infty} \frac{(\frac{1}{2}D + j + l - \sigma |n_4)(\frac{1}{2}D + m |n_1)(-\sigma |n_1)}{n_1! n_4! (1 - i - \sigma - \frac{1}{2}D |n_1 + n_4)} \\ &\times \frac{(1 - i - \frac{1}{2}D |n_4)(1 - i - m - \sigma - D |n_4)}{(1 - i + l - m - \sigma - \frac{1}{2}D |n_4)}, \end{aligned} \quad (21)$$

where

$$\begin{aligned} P_5 &= \frac{\Gamma(\frac{1}{2}D + l - \sigma)\Gamma(1 - i - j - \frac{1}{2}D)}{\Gamma(-j)\Gamma(1 - i - \sigma - \frac{1}{2}D)} \\ &\times \frac{\Gamma(1 - i - m - \sigma - D)}{\Gamma(1 - i + l - m - \sigma - \frac{1}{2}D)}. \end{aligned}$$

Now the sum in n_1 is a ${}_2F_1$ function and we can sum it using the Gauss summation formula (11). The gamma functions that arise from this summation simplify the remaining series in n_4 from a ${}_3F_2$ into a ${}_2F_1$ function that can be summed again using the usual Gauss summation formula. The end result is,

$$\mathcal{J}_b = (-\pi)^D (p^2)^\sigma P_3 P_4 P_5 P_6 P_7, \quad (22)$$

where P_6 came from the n_1 sum,

$$P_6 = \frac{\Gamma(1 - i - \sigma - \frac{1}{2}D)\Gamma(1 - i - m - D)}{\Gamma(1 - i - \frac{1}{2}D)\Gamma(1 - i - m - \sigma - D)},$$

and P_7 came from the last one,

$$P_7 = \frac{\Gamma(-j)\Gamma(1 - i + l - m - \sigma - \frac{1}{2}D)}{\Gamma(\frac{1}{2}D + l - \sigma)\Gamma(1 - \sigma + l)}.$$

Multiplying all the gamma factors (P_3, \dots, P_7) we get, exactly, the expression (14), that leads to the correct result (15). One could rightfully ask: In more complicated problems, the Feynman integrals may not lead to known series. How is one to proceed then? We can answer with an example: the planar three-point two-loop ladder diagram with two external legs on-shell leads to a six indices series, with unit argument, arbitrary exponents of propagators and D . In fact, we can sum five of them and the last one is a ${}_3F_2$ hypergeometric function, which in the special case where the exponents of propagators are minus one reduces to a Gaussian one that is summable, of course. So in the special cases of interest for massless diagrams we will get always products of gamma functions.

For massive internal particles see [7, 9].

3 Special cases

The scalar integral we calculated in the previous section has particular cases of interest, namely, the ‘‘flying saucer’’ diagrams, side view (Fig. 1) and front view (Fig. 2). For the side view diagram the exponents of the propagators are all equal to minus one, while for the front view diagram the exponents are minus one except for $j = -2$.

Let us denote by \mathcal{J}^{AC} the general result for the ‘‘flying saucer’’ diagram; when we take $i = j = l = m = -1$ we have the result for the *side view* diagram,

$$\begin{aligned} \mathcal{J}_{SV}^{AC} &= \pi^D (p^2)^{D-4} \frac{\Gamma^3(\frac{1}{2}D - 1)\Gamma(D - 3)\Gamma(2 - \frac{1}{2}D)}{\Gamma(3 - \frac{1}{2}D)\Gamma(D - 2)} \\ &\times \frac{\Gamma(4 - D)}{\Gamma(\frac{3}{2}D - 4)}, \end{aligned} \quad (23)$$

while when we take $i = -1, j = -2, l = m = -1$ we have the result for the *front view* diagram,

$$\begin{aligned} \mathcal{J}_{FV}^{AC} &= \pi^D (p^2)^{D-5} \frac{\Gamma^3(\frac{1}{2}D - 1)\Gamma(5 - D)\Gamma(D - 4)}{\Gamma(4 - \frac{1}{2}D)\Gamma(D - 2)} \\ &\times \frac{\Gamma(2 - \frac{1}{2}D)}{\Gamma(\frac{3}{2}D - 5)}, \end{aligned} \quad (24)$$

which reproduce well-known results [17, 18].

4 Conclusion

The analytic continuation of the space-time dimension D into *negative values* allowed us to solve two two-loop Feynman integrals with ease. We interpret this analytic continuation like in usual DREG but solve the integrals, in negative dimensions, as they were polynomial. The way back road, via another analytic continuation is straightforward and the whole procedure is quite simple and elegant. There are no cumbersome parametric integrals to solve; on the contrary, the only things one needs to know are how to solve gaussian integrals, and systems of linear algebraic equations! Furthermore, we have surprising manifold degenerate solutions for a single integral. Our previous conjecture on this topic seems to hold, though further research is needed to prove it.

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